17/08/2017 Introduction to Affine Springer fibers ①

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Ref Zhiwei Yun's PCMI lecture notes

[Y] "Lectures on Springer theories & orbital integrals"

& references in there

	field	symmetry	extended symm
Springer fibers	$\mathbb{k}$	W - Weyl gp	graded AHA
affine Springer fibers	local field $F = \mathbb{k}((t))$	$\tilde{W}$ (extended affine)	graded DAHA
Hitch fibers	global field $\mathbb{k}(X)$ ↓ alg curve/$\mathbb{k}$	$\tilde{W}$	graded DAHA

$\mathbb{k} = \mathbb{F}_q$	repn theory
S.F	characters of $G(\mathbb{F}_q)$
A.S.F.	orbital integrals for $G(F)$ (repns of p-adic gps)
H.F	trace formula for G over $\mathbb{k}(X)$

Affine Hecke	DAHA
[K-L, G-C] $H_{aff} \curvearrowright K(\text{Springer fiber})$	$h \curvearrowright K(\text{affine s.f.})$ [Vasserot]
[L] $H_{aff}^{gr} \curvearrowright H^*(\text{Springer fiber})$	$h_\nu^{trig} \curvearrowright H^*(\text{affine s.f})$
	$h_\nu^{rat} \curvearrowright$ [Oblomkov-Yun]

The BLACK BOX theorem

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§1 $\boxed{\text{Loop groups,}}$ parahoric subgps, affine (partial) flag varieties.

$k = \bar{k}$ G: connected reductive gp over k

$F = k((t))$ $\mathcal{O} = k[[t]]$ $\mathrm{val}_F : F^\times \to \mathbb{Z}$ $t \mapsto 1$

positive loop gp : $L^+G : \underset{\mathbb{k}\text{-alg}}{R} \mapsto G(R[[t]])$

repn'able by scheme over $\mathbb{k}$ (not of finite type)

$\boxed{\text{Loop gp:}}$ $LG: \quad R \mapsto G(R((t)))$ repn'able by indscheme $\varinjlim X_m$

e.g. $G = GL_n$ L^+G open subscheme in inf. dim affine space $(a_{ij}^{(s)}, \ i,j \in [1,n] \ s \geq 0)$

LG: $X_m(R) = \{ n \times n$ invertible matrices with entries in $t^{-m}R[[t]] \}$

$\boxed{\underline{\text{Affine Grassmannian}}}$

$$Gr_G(\mathbb{k}) = G(F)/G(\mathcal{O})$$

Let $\mathfrak{g} = \mathrm{Lie}(G)$ $\mathfrak{g}_F = \mathfrak{g} \otimes_{\mathbb{k}} F$ $\mathfrak{g}_{\mathcal{O}} = \mathfrak{g} \otimes_{\mathbb{k}} \mathcal{O}$

The ind-variety Gr_G is an increasing union of finite dimensional projective varieties $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$

As a set $X_n = \{ x \in G(F)/G(\mathcal{O}) \mid \mathrm{Ad}(x^{-1}) \, \mathfrak{g}_{\mathcal{O}} \subset t^{-n} \mathfrak{g}_{\mathcal{O}} \}$

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We have an embedding

$$\phi : X_n \to Gr_n = \text{Grassmannian of subspaces in } t^{-n}\mathfrak{g}_{\mathcal{O}} / t^n\mathfrak{g}_{\mathcal{O}}$$

with $\dim = \dim \mathfrak{g}_{\mathcal{O}}/t^n\mathfrak{g}_{\mathcal{O}}$.

$$x \mapsto Ad(x)\,\mathfrak{g}_{\mathcal{O}}/t^n\mathfrak{g}_{\mathcal{O}} \subset t^{-n}\mathfrak{g}_{\mathcal{O}}/t^n\mathfrak{g}_{\mathcal{O}}$$

This makes X_n an alg. subvar in Gr_n.

<u>Eg:</u> $G = GL_n$

i.e. $\mathcal{O}_F$-submods of F^n of rank n

$$Gr_G(\mathbb{k}) = \{ \mathcal{O}_F\text{-lattices in } F^n \} := \mathcal{L}_n$$

$$g\,G(\mathcal{O}) \mapsto g\Lambda_0 \qquad \Lambda_0 = \mathcal{O}_F^n \text{ standard lattice.}$$

$$X_m = \{ \text{lattices } t^m\mathcal{O}^n \le \Lambda \le t^{-m}\mathcal{O}^n \}$$

$$Gr_G = \varinjlim_m X_m$$

G-simply connected $\qquad T \subset G$ max. torus

$$W = N_G(T)/Z_G(T).$$

$$G(\mathcal{O}) \backslash Gr_G \xrightarrow{\sim} \underbrace{X_*(T)^+}_{\text{dominant cochars}} = X_*(T)/W$$

<u>Cartan decomposition</u>

$$G(F) = \coprod_{\mu \in X_*(T)^+} G(\mathcal{O})\, t^\mu\, G(\mathcal{O})$$

$$\mu : F^\times \to T \qquad t \mapsto t^\mu$$

$Gr_{G,\lambda} = G(\mathcal{O})$-orbit of t^λ $\qquad \overline{Gr_{G,\lambda}}$ - projective

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$\Lambda_1, \Lambda_2 \in \mathcal{L}_n$

$$[\Lambda_1 : \Lambda_2] := \dim_{\mathbb{K}} \Lambda_1 / \Lambda_1 \cap \Lambda_2 - \dim_{\mathbb{K}} \Lambda_2 / \Lambda_1 \cap \Lambda_2$$

$$Gr_{SL_n} = \{ \Lambda \in \mathcal{L}_n \mid [\Lambda : \overset{\mathcal{O}^n}{\overset{\shortparallel}{\Lambda_0}}] = 0 \}.$$

<u>Def</u> 1) Iwahori subgp : Fix $B \subset G$ Borel

Let $\mathbb{I} = \pi^{-1}(B)$ $\quad \pi : G(\mathcal{O}) \to G \quad t \mapsto 0$

e.g. $G = SL_n$ $\quad$ ~~Bx~~ $\mathbb{I} = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ & \ddots & \vdots \\ t\mathcal{O} & & \mathcal{O} \end{pmatrix}$

2) parahoric subgps are connected gp subschemes of LG containing an Iwahori with finite codim

(Precise def: Bruhat-Tits theory)

$\{G(F)\text{-conj classes of parahoric subgps}\}$

$\overset{\sim}{\Longleftrightarrow}$ $\{$proper subsets of vertices of extended Dynkin diagram of $G\}$.

In particular, $G(\mathcal{O}) > \mathbb{I}$ is a parahoric subgp (hyperspecial) e.g. SL_2 $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} = G(\mathcal{O})$

$\begin{pmatrix} \mathcal{O} & t^{-1}\mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix}$ $\quad \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix} = \mathbb{I}$

$\boxed{\text{Affine (partial) flag varieties:}}$ $\quad \mathcal{F}\ell_{\mathbb{P}} = G(F) / \mathbb{P}$

$\mathbb{P} = G(\mathcal{O}) \to$ affine Grassmannian

$\mathbb{P} = \mathbb{I} \to$ affine flag $\quad \mathcal{F}\ell = \mathcal{F}\ell_{\mathbb{I}}$

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e.g. $G = SL_n$.

$Fl = \{$ periodic full chains of lattices

$\cdots \subset \Lambda_{-1} \subset \Lambda_0 \subset \cdots \subset \cdots \quad | \; [\Lambda_i : \mathcal{O}^n] = i$

$\Lambda_i = t\Lambda_{i+n} \}$

§2 Affine Springer fibers. (G – simply connected)

(Concentrate on the case γ regular s.s)

Let $\gamma \in \mathfrak{g}_F$ be regular semisimple (meaning r.s in $\mathfrak{g}_{\bar{F}}$)

Affine Springer fiber of type $\mathbb{P}$ ($\mathbb{P}$ – parahoric)

$\mathcal{X}_{\gamma, \mathbb{P}} := \{ g\mathbb{P} \in Fl_{\mathbb{P}} = G(F)/\mathbb{P} \mid Ad(g^{-1})\gamma \in Lie\,\mathbb{P} \}$

(reduced closed subv. in $Fl_{\mathbb{P}}$)

We write

$\mathcal{X}_\gamma = \mathcal{X}_{\gamma, G(\mathcal{O})}$ $\qquad$ $\mathcal{Y}_\gamma = \mathcal{X}_{\gamma, \mathbb{I}}$

(affine Grassm. version) $\qquad$ (affine flag / Iwahori version).

If $\mathbb{P}_1 \leq \mathbb{P}_2 \Rightarrow \mathcal{X}_{\gamma, \mathbb{P}_1} \to \mathcal{X}_{\gamma, \mathbb{P}_2}$

In particular $\mathcal{Y}_\gamma \to \mathcal{X}_{\gamma, \mathbb{P}}$ (surjective)

First properties (using $\mathcal{X}_\gamma$ as a model)

1) Recall the adjoint quotient map

$\chi : \mathfrak{g} \to \mathfrak{g}/\!/G = \mathfrak{t}/W := \mathfrak{c}$

$\gamma \in \mathfrak{g}_F^{r.s.}$ $\quad \mathcal{X}_\gamma \neq \phi \Leftrightarrow \chi(\gamma) \in \mathfrak{c}^{r.s}(\mathcal{O}_F)$

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2) Let $G_\gamma = Z_{G(F)}(\gamma)$, centralizer of γ in $G(F)$ 17.08.2017

~~γ' This is a~~ γ r.s $\Rightarrow$ G_γ max torus.

G_γ acts on $\mathcal{X}_\gamma$.

$$X_*(G_\gamma) := \mathrm{Hom}_F(\mathbb{G}_m, G_\gamma)$$

$$\Lambda_\gamma := \mathrm{Im}(X_*(G_\gamma) \to G_\gamma(F))$$
$$\lambda \mapsto \lambda(t)$$

<u>Split case</u> $\gamma \in \mathfrak{t}^{rs}(F)$ $G_\gamma = T \otimes_k F$ is F-split

$$G_\gamma(F) \simeq X_*(T) \otimes_{\mathbb{Z}} L\mathbb{G}_m$$

γ is called $\boxed{\text{elliptic}}$ if $X_*(G_\gamma) = \{1\}$.

<u>Thm</u> (Essentially K-L) (These also holds for $\mathcal{X}_{\gamma,\mathbb{P}}$)

$\gamma \in \mathfrak{g}(F)^{rs}$

a) There exists closed subscheme $Z \subset \mathcal{X}_\gamma$, proj. over $\mathbb{k}$

s.t $\mathcal{X}_\gamma = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot Z$

b) $\mathcal{X}_\gamma$ locally of finite type $/\mathbb{k}$.

c) Λ_γ acts freely on $\mathcal{X}_\gamma$ and

$\Lambda_\gamma \backslash \mathcal{X}_\gamma$ is proper over $\mathbb{k}$.

In particular, if γ is elliptic, then $\mathcal{X}_\gamma$ is proper.

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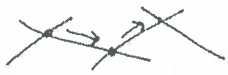
Example I) $G = SL_2$

1) $\gamma = \begin{pmatrix} t & \\ & -t \end{pmatrix}$ $\quad G_\gamma = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in F^\times \right\}$ $\quad X_*(G_\gamma) = \mathbb{Z}$

$\mathcal{X}_\gamma$:  $\cdots\cdots$ ∞-chain of $\mathbb{P}^1$'s

$\xrightarrow[\mathbb{Z}\text{-action}]{}$ $\quad \Lambda \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \Lambda$

$\bigcup_n \{ t^n \mathcal{O} \oplus t^{-n+1}\mathcal{O} \subset \Lambda \subset t^{n-1}\mathcal{O} \oplus t^{-n}\mathcal{O} \}$

$\Lambda_\gamma \backslash \mathcal{X}_\gamma =$ 

2) $\gamma = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ $\quad x \in k^\times$ $\quad \mathcal{X}_\gamma$ discrete set $\cong \mathbb{Z}$

$\quad \mathcal{X}_\gamma = \{ t^{+n}\mathcal{O}_F \oplus t^{-n}\mathcal{O}_F \}$

3) $\gamma = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$

$G_\gamma(F) = \left\{ \begin{pmatrix} a & b \\ bt & a \end{pmatrix} \middle| a, b \in F \quad a^2 - b^2 t = 1 \right\}$ (nonsplit / F)

$\mathcal{X}_\gamma = pt$

4) $\gamma = \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}$ $\quad \mathcal{X}_\gamma = \mathbb{P}^1$

5) $\gamma = \begin{pmatrix} 0 & t^{m+1} \\ t^m & 0 \end{pmatrix}$ $\quad \mathcal{X}_\gamma = \overline{Gr_{\frac{m\check{\alpha}}{2}}}$ $\quad$ m even

$\mathcal{X}_\gamma \cong \overline{Gr^{\frac{m\check{\alpha}}{2}}_{PGL_2}}$ $\quad$ m odd

$\dim \mathcal{X}_\gamma = 2$

Tool $\quad$ Iwasawa decomposition $\quad G(F) = \coprod_{\lambda \in X_*(T)} N(F)\, t^\lambda\, G(\mathcal{O})$

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II) $\gamma \in \mathfrak{t}(\mathcal{O}_F)$ s.t $\bar{\gamma} \in \mathfrak{t}^{rs}$

$\mathcal{X}_\gamma = \{[t^\lambda] \mid \lambda \in X_*(T)\}$

LT acts factoring through L^+T

$\mathcal{X}_\gamma$: Gr_T - torsor.

Dimension formula

split case: $\gamma \in \mathfrak{t}(\mathcal{O}_F)^{rs}$ $\quad \dim \mathcal{X}_\gamma = \sum_{\alpha > 0} val_F \langle \alpha, \gamma \rangle$

In general: $\gamma \in \mathfrak{g}_F^{rs}$ $\quad ad_\gamma : \mathfrak{g}_F \to \mathfrak{g}_F$

induces $\overline{ad_\gamma} : \mathfrak{g}(F)/\mathfrak{g}_\gamma(F) \to \mathfrak{g}(F)/\mathfrak{g}_\gamma(F)$

Define $\Delta(\gamma) = \det(\overline{ad(\gamma)}) \in F^\times$

Thm (Bezrukavnikov)

$$\dim \mathcal{X}_\gamma = \frac{1}{2}(val_F \Delta(\gamma) - c(\gamma))$$

where $c(\gamma) = \operatorname{rank} G - \operatorname{rank} X_*(G_\gamma)$

Idea $\quad \mathcal{X}_\gamma^{reg} := \{g G(\mathcal{O}) \mid Ad(g^{-1})\gamma \mod t \in \mathfrak{g}(k)^{reg}\}$

$\cap$ open

$\mathcal{X}_\gamma \qquad \mathcal{X}_\gamma^{reg} \neq \phi$

$\dim \mathcal{X}_\gamma^{reg} \overset{\dim}{=} \mathcal{X}_\gamma \qquad$ (Ngô: $\mathcal{X}_\gamma^{reg}$ dense in $\mathcal{X}_\gamma$)

$G_\gamma(F) \circlearrowright \mathcal{X}_\gamma^{reg}$ transitively

$\mathcal{X}_\gamma^{reg} = G_\gamma / \text{"cpt" open subgp} = $ f.d. comm. gp $/k$ (possibly ∞ many components)

Further more

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- $\boxed{\text{[Ngô]}}$ LG_γ action factors through

spectral curve → $a = \chi(\gamma) \in C$ $P_a := LG_\gamma / L^+ J_a$ (local Picard gp)

and does not factor through any further quotient.

(J_a is an integral model of G_γ over $\mathcal{O}$)

$$\begin{array}{ccc} & & I \\ & & \downarrow \\ \chi^* J & \to & \mathfrak{g} \\ \downarrow & & \downarrow \chi \\ J & \to & C \end{array} \qquad \begin{array}{ccc} J_a & \to & \operatorname{spec} \mathcal{O}_F \\ \downarrow & & \downarrow \\ J & \to & C \end{array}$$

iso over $\mathfrak{g}^{reg}$: I → universal centralizer gp scheme

Regular centralizer gp scheme ← J

- $\boxed{\text{[Ngô]}}$ Each irr comp of $\mathcal{X}_\gamma$ is a rational variety.

This rational property <u>fails</u> for general $\mathcal{X}_{\gamma, \mathbb{P}}$.

(e.g. Bernstein - Kazhdan example)

- Purity conjecture (GKM) is open. in general.

GKM: γ equivalued, coho of affine S.f is pure.

- [Affine Springer repns] (Lusztig, Sage)

$$\widetilde{W} = X_*(T) \rtimes W$$

There is a canonical action of $\widetilde{W}$

on $H_*(\mathcal{Y}_\gamma)$.

↓

Iwahori version.

$$(H_*(\mathcal{Y}_\gamma) := \varinjlim_n H_*(\mathcal{Y}_{\gamma,n}))$$

$1 \to \mathbb{P}^+ \to \mathbb{P} \to L_{\mathbb{P}} \to 1$

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sketch from [Y]

$$\begin{tikzcd} Y_\gamma \arrow[r, "ev_{I,\gamma}"] \arrow[d, "\pi_{\mathbb{P},\gamma}"'] & \left[b_{\mathbb{P}}^{I} / B_{\mathbb{P}}^{I} \right] \arrow[d, "\pi_{\ell_{\mathbb{P}}}"] \\ \mathcal{X}_{\gamma,\mathbb{P}} \arrow[r, "ev_{\mathbb{P},\gamma}"'] \arrow[ur, phantom, "\lrcorner"] & \left[\ell_{\mathbb{P}} / L_{\mathbb{P}} \right] \end{tikzcd}$$

$\rightsquigarrow$ Grothendieck's simultineous res.

$[g] \mapsto \pi_{\mathbb{P}}(Ad(g^{-1})\gamma)$

$\pi_{\mathbb{P}}: \mathrm{Lie}\,\mathbb{P} \to \mathrm{Lie}\,L_{\mathbb{P}} = \ell_{\mathbb{P}}$

$b_{\mathbb{P}}^{I} = \pi_{\mathbb{P}}(\mathrm{Lie}\,I)$

$B_{\mathbb{P}}^{I} = \mathrm{Im}(I)$ under $\mathbb{P} \to L_{\mathbb{P}}$

Classical Springer

W_{L_P} acts on $\pi_{\ell_P *} \mathbb{D}$ (dualizing cpx)

base change $\Rightarrow$ W_{L_P} acts on $\pi_{\mathbb{P},\gamma *} \mathbb{D}$, thus acts on $H_*(Y_\gamma)$

Take P, L_P s.t. $W_{L_P} = \langle s_i \rangle \Rightarrow s_i$ acts

To check braid ~~rels~~ relations, take $\mathbb{P}$, s.t $W(L_P) = \langle s_i, s_j \rangle$.

- This action is not necess. s.s.

e.g. $\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$ $\quad G = SL_2$

$Y_\gamma = C_0 \cup C_1 \qquad H_2(Y_\gamma) = \langle [C_0], [C_1] \rangle$

$s_0 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad s_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

$1 \to \text{trivial} \to H_2(Y_\gamma) \to \text{sgn} \to 1$

nontrivial ext.

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(11)

- This action can be extended to

$Sym(X^*(T)) \rtimes \tilde{W}$ The BLACK BOX theorem

$$\boxed{\text{Graded DAHA } \circlearrowright H^*_{\mathbb{G}_m}(\mathcal{Y}_\gamma) \quad (\text{O-Y})}$$
$\hookrightarrow$ homogeneous

§3 Hessenberg pavings for homogeneous affine Springer fibers. (Oblomkov-Yun) $k = \mathbb{C}$

Let $\boxed{\nu = \frac{d}{m} > 0}$

Def An elt $\gamma(t) \in \mathfrak{g}_F^{rs}$ is homogeneous of slope ν if

$$\forall s \in k^\times, \quad \gamma(s^m t) \underset{conj}{\sim} s^d \gamma(t).$$

e.g.: $\begin{pmatrix} at^2 & \\ & -at^2 \end{pmatrix}$ slope 2 $\begin{pmatrix} 0 & 1 \\ t^3 & 0 \end{pmatrix}$ slope $\frac{3}{2}$.

let $\mathbb{G}_m(\nu)$ be the one-diml torus acting on $G(F)$ as follows: $\check{\rho}: \mathbb{G}_m \to G$

$$s \in k^\times \qquad s \cdot_\nu g(t) = Ad(s^{d\check{\rho}})\, g(s^m t)$$

Let

$$\mathfrak{g}(F)_\nu = \{\gamma \in \mathfrak{g}(F) \mid \gamma(s^m t) = Ad(s^{-d\check{\rho}})(s^d \gamma(t)) \quad \forall s \in k^\times\}$$

(wt space with wt d under $\mathbb{G}_m(\nu)$)

$\mathfrak{g}(F)_\nu$: finite diml over k (a graded-piece in the Moy-Prasad filtration for $\mathfrak{g}(F)$)

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It includes all homogeneous elt of slope ν up to conjugacy by $G(F)$. $[\mathfrak{g}_\nu \cap \mathfrak{g}_F^{rs} \neq \phi \Leftrightarrow m$ regular number$]$

$\nu \to$ Parahoric subgp $\mathbb{P}_\nu$ with Levi quotient L_ν

$W_\nu \subset \widetilde{W}$ Weyl gp of L_ν

Def Hessenberg variety.

M: reductive gp $/ \mathbb{C}$ $\quad$ V: linear repn of M

$P \subset M$ parabolic $\quad$ $V^+ \subset V$ P-stable subspace

$$\begin{array}{l} M \overset{P}{\times} V^+ \\ \downarrow \pi \\ V \end{array} \qquad \begin{array}{l} v \in V \\ \mathrm{Hess}_v(M/P,\ V^+ \subset V) := \pi^{-1}(v) \\ = \{gP \in M/P \mid g^{-1}.v \in V^+\} \subset M/P \end{array}$$

Consider the family

$$\widetilde{\mathcal{Z}}_{\mathbb{P},\nu}: \quad \begin{array}{c} \widetilde{\mathcal{X}}_{P,\nu} \\ \downarrow \\ \mathfrak{g}(F)_\nu^{rs} \end{array}$$

For each $\tilde{w} \in \widetilde{W}$

$$\mathrm{Hess}_{\mathbb{P},\gamma}^{\tilde{w}} := \mathrm{Hess}_\gamma\left(L_\nu / L_\nu \cap \mathrm{Ad}(\tilde{w})\mathbb{P},\ \mathfrak{g}(F)_{\mathbb{P},\nu}^{\tilde{w}} \subset \mathfrak{g}(F)_\nu \right)$$

$$\mathfrak{g}(F)_{\mathbb{P},\nu}^{\tilde{w}} = \mathfrak{g}(F)_\nu \cap \mathrm{Ad}(\tilde{w}) \,\mathrm{Lie}\,\mathbb{P}$$

(only depends on the class of $\tilde{w}$ in $W_\nu \backslash \widetilde{W} / W_{\mathbb{P}}$)

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Family

$$\begin{array}{ccc} \widetilde{\mathrm{Hess}}^{\tilde w}_{\mathbb{P},\nu} & & \mathrm{Hess}^{\tilde w}_{\mathbb{P},\gamma} \\ \downarrow \tilde\pi^{\tilde w}_{\mathbb{P},\nu} & & \vdots \\ \mathfrak{g}(F)_\nu & & \gamma \end{array}$$

<u>Thm</u> (GKM)

(1) $$\widetilde{\mathcal{X}}^{\mathbb{G}_m(\nu)}_{\mathbb{P},\nu} = \coprod_{\tilde w \in W_\nu \backslash \widetilde{W} / W_{\mathbb{P}}} \widetilde{\mathrm{Hess}}^{\tilde w}_{\mathbb{P},\nu} \Big|_{\mathfrak{g}(F)^{rs}_\nu}$$

(2) $\gamma \in \mathfrak{g}(F)^{rs}_\nu$

$\mathcal{X}_{\mathbb{P},\gamma}$ admits a pavement by intersecting with $\mathbb{P}_\nu$ - orbits in $\mathcal{F}\ell_{\mathbb{P}}$.

Each intersection $(\mathbb{P}_\nu \tilde w \mathbb{P}/\mathbb{P}) \cap \mathcal{X}_{\mathbb{P},\gamma}$ is an affine space bundle over $\mathrm{Hess}^{\tilde w}_{\mathbb{P},\cancel{\nu},\gamma}$ which contracts to $\mathrm{Hess}^{\tilde w}_{\mathbb{P},\gamma}$ under $\mathbb{G}_m(\nu)$ - action

(3) $\tilde\pi^{\tilde w}_{\mathbb{P},\nu}$ is smooth over $\mathfrak{g}(F)^{rs}_\nu$.

$\widetilde{\mathrm{Hess}}^{\tilde w}_{\mathbb{P},\nu} \to \mathfrak{g}(F)_\nu$

(4) Cohomology of $\mathcal{X}_{\mathbb{P},\gamma}$ is pure. $\gamma \in \mathfrak{g}(F)^{rs}_\nu$.

$H^*_{\mathbb{G}_m(\nu)}(\mathcal{Y}_\gamma)$ mod $\mathbb{C}[\varepsilon] = H^*_{\mathbb{G}_m}(pt)$

[O-Y] $$\mathrm{Gr}^P_* \, H^*_{\varepsilon=1}(\cancel{\mathcal{X}}\ \mathcal{Y}_\gamma)^{S_\nu \rtimes B_\nu} = L_\nu \text{ (trivial) for } \mathfrak{h}^{rat}_\nu.$$

P: perverse filtration $\quad S_\nu = \mathrm{Stab}_{L_\nu}(\gamma) \quad B_\nu$: Braid gp attached to certain cpx refl. gp

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- Affine Bruhat decomposition

$$\mathcal{F}l_{\mathbb{P}} = \coprod_{\tilde{w} \in W_\nu \backslash \tilde{W} / W_{\mathbb{P}}} \mathbb{P}_\nu \tilde{w} \mathbb{P}/\mathbb{P}.$$

$$\mathcal{F}l_{\mathbb{P}}^{\mathbb{G}_m(\nu)} = \coprod_{\tilde{w} \in W_\nu \backslash \tilde{W} / W_{\mathbb{P}}} L_\nu \tilde{w} \mathbb{P}/\mathbb{P}$$

- L_ν is the identity component of $\tilde{L}_\nu = G(F)^{\mathbb{G}_m(\nu)}$